MULTIPLE SOLUTIONS TO SINGULAR FOURTH ORDER ELLIPTIC EQUATIONS

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Abstract. Using the method of Nehari manifold, we prove the existence of at least two distinct weak solutions to elliptic equation of four order with singulatities and with critical Sobolev growth.

1. Introduction

Fourth order elliptic equations have been intensively investigated the last tree decades particularly after the discovery of an important conformally invariant operator by Paneitz on 4 - dimensional Riemannian manifolds [19] and whose definition was extended to higher dimension by Branson [8].This operator is closely related to the problem of prescribed Q- curvature. Many works have been devoted to this subject (see [1], [2], [3], [4], [5], [6], [7], [11], [12], [13], [14], [15], [16], [17], [20], [21], [22], [23], [24]). Let (M,g) a compact smooth Riemannian of dimension $n \geq 5$ with a metric g. We denote by $H_2^2(M)$ the standard Sobolev space which is the completion of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{2,2} = \sum_{k=0}^{k=2} \left\|\nabla^k \varphi\right\|_2 .$$

$H_2^2(M)$ will be endowed with the equivalent suitable norm

$$\|u\|_{H_2^2(M)} = (\int_M \left((\Delta_g u)^2 + |\nabla_g u|^2 + u^2\right) dv_g)^{\frac{1}{2}}.$$

Recently, Madani [18], has considered the Yamabe problem with singularities which he solved under some geometric conditions. The first author in [6] considered singular fourth order elliptic equations with of the form

$$\Delta^2 u - \nabla^i \left(a(x)\nabla_i u\right) + b(x)u = f\,|u|^{N-2}\,u \tag{1.1}$$

where the functions a and b are in $L^s(M)$, $s > \frac{n}{2}$ and in $L^p(M)$, $p > \frac{n}{4}$ respectively, $N = \frac{2n}{n-4}$ is the Sobolev critical exponent in the embedding $H_2^2(M) \hookrightarrow L^N(M)$. He established the following result. Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in$

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$L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $x_o \in M$ such that $f(x_o) = \max_{x\in M} f(x)$.

Theorem 1. *Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $P \in M$ such that $f(P) = \max_{x\in M} f(x)$.*

For $n \geq 10$,or $n = 9$ and $\frac{9}{4} < p < 11$ or $n = 8$ and $2 < p < 5$ or $n = 7$ and $\frac{7}{2} < s < 9$, $\frac{7}{4} < p < 3$ we suppose that

$$\frac{n^2+4n-20}{6(n-6)(n^2-4)} R_g(P) - \frac{n-4}{2n(n-2)} \frac{\Delta f(P)}{f(P)} > 0.$$

For $n = 6$ and $\frac{3}{2} < p < 2$, $3 < s < 4$, we suppose that

$$R_g(P) > 0.$$

Then the equation (1.1) has a non trivial weak solution u in $H_2^2(M)$. Moreover if $a \in H_1^s(M)$, then
$u \in C^{0,\beta}$, for some $\beta \in \left(0, 1 - \frac{n}{4p}\right)$..

For fixed $R \in M$, we define the function ρ on M by

$$\rho(Q) = \begin{cases} d(R,Q) & \text{if} \quad d(R,Q) < \delta(M) \\ \delta(M) & \text{if} \quad d(R,Q) \geq \delta(M) \end{cases} \tag{1.2}$$

where $\delta(M)$ denotes the injectivity radius of M.

In this paper, we are concerned with the following problem: for real numbers σ and μ, consider the equation in the distribution sense

$$\Delta^2 u - \nabla^i(a\rho^{-\mu}\nabla_i u) + \rho^{-\alpha} b u = \lambda |u|^{q-2} u + f(x) |u|^{N-2} u \tag{1.3}$$

where the functions a and b are smooth M and $1 < q < 2$. Denote also by P_g the operator define on $H_2^2(M)$ by $u \to P_g(u) = \Delta^2 u - \nabla^i(a\rho^{-\mu}\nabla_i u) + \rho^{-\alpha} b u$. Our main results state as follows:

Theorem 2. *Let $0 < \sigma < 2$ and $0 < \mu < 4$. Suppose that the operator P_g is coercive and*
(C)

$$\begin{cases} \frac{\Delta f(x_o)}{f(x_o)} < \frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{(n^2-4)(n-4)(n-6)} \frac{1}{\left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r + \left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{4}{n}}} - 1\right) S_g(x_o) \ \text{ in case } n > 6 \\ S_g(x_\circ) > 0 \quad \text{in case } n = 6. \end{cases}$$

Then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the equation (1.3) possesses at least two distinct non trivial solutions in the distribution sense.

The proof of Theorem 2 relies on the following Hardy-Sobolev inequality (see [4]).

Lemma 1. *Let (M,g) be a compact n- dimensional Riemannian manifold, and p, q and γ real numbers satisfying*

$$1 \leq q \leq p \leq \frac{nq}{n-2q},\ n > 2q,\ \frac{\gamma}{p} = -2 + n\left(\frac{1}{q} - \frac{1}{p}\right) > -\frac{n}{p}.$$

For any $\varepsilon > 0$, there is a constant $A(\varepsilon, q, \gamma)$ such that

$$\forall f \in H_2^q(M), \quad \|f\|_{p,\rho^\gamma}^q \leq (1+\varepsilon)\, K^q(n,q,\gamma) \left\|\nabla^2 f\right\|_q^q + A(\varepsilon,q,\gamma)\, \|f\|_q^q .$$

In particular in case $\gamma = 0$, $K(n,q,0) = K(n,q)$ is the best constant in Sobolev's inequality.

For brevity along all this work we put $K_o = K(n,2)$.

Let σ and μ be as in Theorem 2, the Hardy- Sobolev inequality given by Lemma 1 leads to

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(\|\nabla |\nabla u|\|^2 + \|\nabla u\|^2)$$

and since

$$\|\nabla |\nabla u|\|^2 \leq \left\|\nabla^2 u\right\|^2 \leq \|\Delta u\|^2 + \beta \|\nabla u\|^2$$

where $\beta > 0$ is a constant and it is well known that for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u\|^2 \leq \varepsilon \|\Delta u\|^2 + c\, \|u\|^2 .$$

Hence

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C\,(1+\varepsilon)\, \|\Delta u\|^2 + A\,(\varepsilon,\sigma)\, \|u\|^2 . \tag{1.4}$$

Let $K(n,2,\sigma)$ be the best constant in inequality (1.4) and $K(n,2,\mu)$ the best one in the inequality

$$\int_M \frac{|u|^2}{\rho^\mu} dv_g \leq C\,(1+\varepsilon)\, \|\Delta u\|^2 + A\,(\varepsilon,\mu)\, \|u\|^2 .$$

For any $0 < \sigma < 2$ and $0 < \mu < 4$, let $u_{\sigma,\mu}$ be the solution of Equation (1.7). In the sharp case $\sigma = 2$ and $\mu = 4$, we obtain the following result

Theorem 3. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Suppose that the operator P_g is coercive and let $\left(u_{\sigma,\mu}\right)_{\sigma,\mu}$ be a sequence in M_λ such that*

$$\begin{cases} J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) \leq c_{\sigma,\mu} \\ \nabla J_\lambda(u_{\sigma,\mu}) - \mu_{\sigma,\mu} \nabla \Phi_\lambda(u_{\sigma,\mu}) \to 0 \end{cases} .$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n\ K_o^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max\left(K(n,2,\sigma), A\,(\varepsilon,\sigma)\right) + b^- \max\left(K(n,2,\mu), A\,(\varepsilon,\mu)\right) > 0$$

then the equation

$$\Delta^2 u - \nabla^{\mu}(\frac{a}{\rho^2}\nabla_{\mu} u) + \frac{bu}{\rho^4} = f\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u$$

has at least two distinct non trivial solutions in distribution sense.

We consider the energy functional J_λ defined by for each $u \in H_2^2(M)$ by

$$J_\lambda(u) = \frac{1}{2}\int_M \left((\Delta_g u)^2 - a(x)\rho^{-\sigma}\,|\nabla_g u|^2 + b(x)\rho^{-\mu}u^2\right) dv(g) - \frac{\lambda}{q}\int_M |u|^q\, dv(g)$$
$$-\frac{1}{N}\int_M f(x)\,|u|^N\, dv(g).$$

Put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u),\ u\rangle$$

$$\Phi_\lambda(u) = \int_M \left((\Delta_g u)^2 - a(x)\rho^{-\sigma}\,|\nabla_g u|^2 + b(x)\rho^{-\mu}u^2\right) dv(g) - \lambda\int_M |u|^q\, dv(g)$$
$$-\int_M f(x)\,|u|^N\, dv(g)$$

and

$$\langle \nabla\Phi_\lambda(u), u\rangle = 2\int_M \left((\Delta_g u)^2 - a(x)\rho^{-\sigma}\,|\nabla_g u|^2 + b(x)\rho^{-\mu}u^2\right) dv(g) - \lambda q\int_M |u|^q\, dv(g)$$
$$-\lambda q\int_M |u|^q\, dv(g) - N\int_M f(x)\,|u|^N\, dv(g).$$

It is well-known that the solutions of equation (1.3) are critical points of the energy functional J_λ. The Nehari minimization problem writes as follows

$$\alpha_\lambda = \inf\ \ \{J_\lambda(u) : u \in N_\lambda\} = \inf_{u\in N_\lambda} J_\lambda(u)$$

where

$$N_\lambda = \left\{u \in H_2^2(M)\backslash\{0\} :\ \Phi_\lambda(u) = 0\right\}.$$

Note that N_λ contains every solution of equation (1.3).
N_λ splits in three parts

$$N_\lambda^+ = \{u \in N_\lambda :\ \ \langle\nabla\Phi_\lambda(u),\ u\rangle > 0\}$$
$$N_\lambda^- = \{u \in N_\lambda :\ \ \langle\nabla\Phi_\lambda(u),\ u\rangle < 0\}$$
$$N_\lambda^0 = \{u \in N_\lambda :\ \ \langle\nabla\Phi_\lambda(u),\ u\rangle = 0\}.$$

Before stating our main result, we give some nice properties of N_λ^+, N_λ^- and N_λ^0.
Let

$$\lambda_\circ = \frac{(N-2)\,q\,\Lambda^{\frac{q}{2}}}{2\,(N-q)\,V(M)^{1-\frac{q}{N}}(\max(K_\circ, A_\varepsilon))^{\frac{q}{2}}} \tag{1.5}$$

The following lemma shows that the minimizers of J_λ on N_λ are usually critical points for J_λ.

Lemma 2. *Let $\lambda \in (0,\ \lambda_\circ)$, if v is a local minimizer for J_λ on N_λ and $v \notin N_\lambda^0$, then $\nabla J_\lambda(v) = 0$.*

Proof. If v is a local minimizer for $J_\lambda$on N_λ, then by Lagrange multipliers' theorem, there is a real θ such that for any $\varphi \in H_2^2(M)$

$$\langle \nabla J_\lambda(v), \varphi \rangle = \theta \langle \nabla \Phi_\lambda(v), \varphi \rangle$$

If $\theta = 0$, then the lemma is proved. If it is not the case we pick $\varphi = v$ and we use the assumption that $v \in N_\lambda$ to infer that

$$\langle \nabla J_\lambda(v), v \rangle = \theta \langle \nabla \Phi_\lambda(v), v \rangle = 0$$

which contradicts that $v \notin N_\lambda^0$. □

Now we give some preparatory lemmas.

Lemma 3. *There is $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ the set N_λ^0 is empty .*

Proof. Suppose for every $\lambda > 0$ there is $\lambda' \in (0, \lambda)$ such that $N_{\lambda'}^0, \neq \emptyset$ and let $u \in N_{\lambda'}^0$ i.e.

$$\langle \nabla \Phi_{\lambda'}(u), u \rangle = 2 \|u\|^2 - \lambda' q \|u\|_q^q - N \int_M f(x) |u|^N dv(g) = 0$$

and by the fact that

$$\Phi_{\lambda'}(u) = \|u\|^2 - \lambda' \|u\|_q^q - \int_M f(x) |u|^N dv(g) = 0$$

we get

$$\|u\|^2 = \frac{N-q}{2-q} \int_M f(x) |u|^N dv(g) \tag{1.6}$$

and also

$$\lambda' \|u\|_q^q = \frac{N-2}{2-q} \int_M f(x) |u|^N dv(g). \tag{1.7}$$

Independently by the Sobolev's inequality and the coerciveness of the operator P_g we obtain

$$\int_M f(x) |u|^N dv(g) \leq \Lambda^{-\frac{N}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \max_{x \in M} f(x) \|u\|^N \tag{1.8}$$

where Λ denotes a constant of the coercivity. From (1.6) and (1.8) we deduce that

$$\|u\| \geq \left[\frac{(N-q)\, \Lambda^{-\frac{N}{2}} \left((\max((1+\varepsilon)K_\circ, A_\varepsilon)\right)^{\frac{N}{2}} \max_{x \in M} f(x)}{(2-q)} \right]^{\frac{1}{2-N}}$$

Let the functional $I_{\lambda'} : N_\lambda \to \mathbb{R}$ is given by

$$I_{\lambda'}(u) = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-2} \right]^{\frac{2}{2-q}} \left(\frac{\|u\|^q}{\lambda' \|u\|_q^q} \right)^{\frac{2}{q-2}} - \int_M f(x) |u|^N dv(g).$$

If $u \in N^0_{\lambda'}$, then (1.6) and (1.7) give

$$I_{\lambda'}(u) = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-2} \right]^{\frac{2}{2-q}} \left[\frac{\left(\frac{N-q}{2-q} \int_M f(x) \left| u \right|^N dv(g) \right)^{\frac{q}{2}}}{\frac{N-2}{2-q} \int_M f(x) \left| u \right|^N dv(g)} \right]^{\frac{2}{q-2}}$$

$$- \int_M f(x) \left| u \right|^N dv(g) = 0. \tag{1.9}$$

Putting

$$\theta = \left[\left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \frac{2-q}{N-2} \right]^{\frac{2}{2-q}}$$

and taking account of the coerciveness of the operator P_g and the Sobolev's inequality one get

$$I_{\lambda'}(u) \geq \theta \left(\frac{\|u\|^q}{\lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q} \right)^{\frac{2}{q-2}}$$

$$-\Lambda^{-\frac{N}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{N}{2}} \max_{x \in M} f(x) \|u\|^N .$$

That is to say

$$I_{\lambda'}(u) \geq \left(\frac{\Lambda^{\frac{q}{2}} \left(\frac{N-q}{2-q} \right)^{\frac{q}{2}} \left(\frac{2-q}{N-2} \right) \left(\frac{Nq}{N-q} \right)}{\lambda' V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} \right)^{\frac{2}{q-2}}$$

$$- \left(\left(\frac{N-q}{2-q} \right) \Lambda^{-\frac{N}{2}} \left((\max((1+\varepsilon)K_\circ, A_\varepsilon) \right)^{\frac{N}{2}} \max_{x \in M} f(x) \right)^{\frac{2}{2-N}} .$$

Hence, if λ is sufficiently small, so as $\lambda' > 0$ and $I_{\lambda'}(u) > 0$ for all $u \in N^0_{\lambda'}$. This contradicts (1.9). So there is $\lambda_1 > 0$, such that for any $\lambda \in (0, \lambda_1)$, the set $N^0_\lambda = \emptyset$. □

From Lemma 3, N_λ splits as $N_\lambda = N^+_\lambda \cup N^-_\lambda$ where $0 < \lambda < \lambda_1$. We define

$$\alpha_\lambda = \inf_{u \in N_\lambda} J_\lambda(u), \quad \alpha^+_\lambda = \inf_{u \in N^+_\lambda} J_\lambda(u) \quad \text{and} \quad \alpha^-_\lambda = \inf_{u \in N^-_\lambda} J_\lambda(u)$$

Lemma 4. *For each $\lambda \in (0, \lambda_\circ)$, the functional J_λ is bounded from below on N_λ.*

Proof. If $u \in N_\lambda$,then by equality (1.6) and the Sobolev's inequality, we deduce that

$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|u\|^q_{H^2_2(M)}$$

and taking account of the coerciveness of the operator P_g, we infer that

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^q$$

where Λ is a constant of coercivity.

If $u \in N_\lambda$ and $\|u\| \geq 1$,

$$J_\lambda(u) \geq \left[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\right]\|u\|^q$$

So, if

$$0 < \lambda < \frac{(N-2)\,q\,\Lambda^{\frac{q}{2}}}{2\,(N-q)\,V(M)^{1-\frac{q}{N}}(\max(K_\circ, A_\varepsilon))^{\frac{q}{2}}} := \lambda_\circ$$

then

$$J_\lambda(u) > 0$$

If $u \in N_\lambda$ with$\|u\| < 1$, we have

$$J_\lambda(u) > -\lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}.$$

Thus J_λ is bounded below on N_λ . □

As a consequence of Lemma 2 we have

Lemma 5. *If* $\lambda \in (0, \lambda_\circ)$, *we have*

$$\alpha_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u) < 0.$$

Proof. If $u \in N_\lambda^+$, then

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|^2 - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

and since

$$\langle \nabla\Phi_\lambda(u), u\rangle = 2\|u\|^2 - \lambda q\|u\|_q^q - N\int_M f(x)\,|u|^N\,dv(g) > 0$$

we get

$$J_\lambda(u) \leq \frac{\lambda(N-q)}{N}\left(\frac{1}{2} - \frac{1}{q}\right)\|u\|_q^q < 0$$

i.e.

$$\inf_{u \in N_\lambda^+} J_\lambda(u) < 0.$$

□

Lemma 6. *For every* $\lambda \in (0, \min(\lambda_0, \lambda_1))$,

$$\alpha_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u) > 0.$$

Proof. If $u \in N_\lambda^-$, then

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|^2 - \frac{\lambda(N-q)}{Nq}\|u\|_q^q$$

and since

$$\langle \nabla \Phi_\lambda(u), u\rangle = 2\|u\|^2 - \lambda q \|u\|_q^q - N\int_M f(x)|u|^N dv(g) < 0 \tag{1.10}$$

we infer that

$$\|u\|^2 > \frac{\lambda(N-q)}{(N-2)}\|u\|_q^q . \tag{1.11}$$

By Sobolev's inequality and from the coerciveness of the operator P_g, there exists a constant $\Lambda > 0$, such that

$$J_\lambda(u) \geq \frac{N-2}{2N}\|u\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\|u\|^q .$$

So if $u \in N_\lambda^-$ and $\|u\| \geq 1$,

$$J_\lambda(u) \geq \left[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}\right]\|u\|^q \tag{1.12}$$

hence if

$$0 < \lambda < \frac{(N-2)\,q\,\Lambda^{\frac{q}{2}}}{2\,(N-q)\,V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}} = \lambda_\circ$$

then

$$J_\lambda(u) > 0$$

In case $u \in N_\lambda^-$ and $\|u\| < 1$, by Sobolev's inequality, the inequality (1.10) and the coerciveness of the operator P_g, we obtain

$$0 < \xi \leq \|u\| < 1$$

where

$$\xi = \left[\frac{(2-q)\,\Lambda^{\frac{N}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{N}{2}}}{(N-q)\max_{x\in M} f(x)}\right]^{\frac{1}{N-2}}$$

and Λ is a constant of coerciveness.

The inequality (1.12) becomes

$$J_\lambda(u) \geq \frac{N-2}{2N}\xi^2 - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}}$$

Hence, if we take

$$0 < \lambda < \frac{\frac{(N-2)}{2(N-q)}\xi^2\Lambda^{\frac{q}{2}}}{V(M)^{1-\frac{q}{N}}(\max(K_\circ, A_\varepsilon))^{\frac{q}{2}}} = \lambda_2 \tag{1.13}$$

then if $\lambda \in (0, \min(\lambda_0, \lambda_1, \lambda_2))$ we obtain

$$J_\lambda(u) \geq C > 0$$

where C is constant depending on N, Λ, $V(M)$, K_o and A_ε. So

$$\inf_{u\in N_\lambda^-} J_\lambda(u) > 0.$$

□

For each $u \in H_2 - \{0\}$, define

$$E(t) = t^{2-q}\left\|u\right\|^2 - t^{N-q}\int_M f\left|u\right|^N dv_g$$

so $E(0) = 0$ and $E(t)$ goes to $-\infty$ as $t \to +\infty$. Also for $t > 0$, we have

$$E'(t) = (2-q)\,t^{1-q}\left\|u\right\|^2 - (N-q)t^{N-q-1}\int_M f\left|u\right|^N dv_g$$

and $E'(t) = 0$ at

$$t_o = \left(\frac{2-q}{N-q}\right)^{\frac{1}{N-2}}\left(\frac{\left\|u\right\|^2}{\int_M f\left|u\right|^N dv_g}\right)^{\frac{1}{N-2}}.$$

Hence $E(t)$ achieves its maximum at t_o and it is increasing on $[0, t_o)$ and decreasing on $[t_o, +\infty)$.

Evaluating the function E at t_o,

$$E(t_o) = \left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-2}}\left(\frac{\left\|u\right\|^2}{\int_M f\left|u\right|^N dv_g}\right)^{\frac{2-q}{N-q}}\left\|u\right\|^2$$

$$-\left(\frac{2-q}{N-q}\right)^{\frac{N-q}{N-2}}\left(\frac{\left\|u\right\|^2}{\int_M f\left|u\right|^N dv_g}\right)^{\frac{N-q}{N-2}}\int_M f\left|u\right|^N dv_g$$

$$= \frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-2}}\frac{\left\|u\right\|^{\frac{2(N-q)}{N-2}}}{\left(\int_M f\left|u\right|^N dv_g\right)^{\frac{2-q}{N-2}}}.$$

By the Sobolev's inequality we get for any $\epsilon > 0$,

$$\int_M f\left|u\right|^N dv_g \le \left\|f\right\|_\infty\left(\left(K_o^2+\epsilon\right)\left\|\Delta u\right\|_2^2 + A\left(\epsilon\right)\left\|u\right\|_2^2\right)^{\frac{N}{2}}$$

$$\le \left\|f\right\|_\infty \max\left(K_o^2+\epsilon, A\left(\epsilon\right)\right)^{\frac{N}{2}}\left\|u\right\|_{H_2^2}^N$$

$$\le \Lambda^{-\frac{N}{2}}\left\|f\right\|_\infty \max\left(K_o^2+\epsilon, A\left(\epsilon\right)\right)^{\frac{N}{2}}\left\|u\right\|^N$$

$$= C^{\frac{N}{2}}\left\|f\right\|_\infty\left\|u\right\|^N$$

where Λ is the constant of the coercivity, K_o the best constant in the Sobolev's inequality and $A\left(\epsilon\right)$ the correspondent constant, $\left\|f\right\|_\infty = \sup_{x\in M}\left|f(x)\right|$ and $C = \Lambda^{-1}\max\left(K_o^2+\epsilon, A\left(\epsilon\right)\right)$.

Consequently

$$E\left(t_o\right)\geq\frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-q}}C^{\frac{N(q-2)}{2(N-2)}}\left\Vert f\right\Vert_{\infty}\left\Vert u\right\Vert^{q}. \tag{1.14}$$

Independently and in the same way as above we get

$$\left\Vert u\right\Vert_{q}^{q}\leq\Lambda^{-\frac{q}{2}}vol(M)^{1-\frac{q}{N}}C^{\frac{q}{2}}\left\Vert u\right\Vert^{q}. \tag{1.15}$$

Hence

$$E(0)=0<\lambda\left\Vert u\right\Vert_{q}^{q}\leq E(t_o)$$

provided that

$$\lambda\leq\frac{\frac{N-2}{N-q}\left(\frac{2-q}{N-q}\right)^{\frac{2-q}{N-q}}\left\Vert f\right\Vert_{\infty}}{vol(M)^{1-\frac{q}{2}}C^{\frac{N-q}{N-2}}}.$$

Consequently by the nature of the function $E(t)$ we infer the existence of t^{-}, t^{+} with $0<t^{+}<t^{o}<t^{-}$ such that

$$\lambda\left\Vert u\right\Vert_{q}^{q}=E(t^{+})=E(t^{-}). \tag{1.16}$$

and

$$E'(t^{+})>0>E'(t^{-})$$

Now we evaluate Φ_λ at $t^{-}u$ and at $t^{+}u$ to get

$$\begin{aligned}\Phi_\lambda(t^{-}u)&=\left\langle\nabla J_\lambda\left(t^{-}u\right),t^{-}u\right\rangle\\&=\left(t^{-}\right)^{2}\left\Vert u\right\Vert^{2}-\left(t^{-}\right)^{N}\int_M f\left\vert u\right\vert^{N}dv_g-\lambda\left(t^{-}\right)^{q}\left\Vert u\right\Vert_{q}^{q}\\&=\left(t^{-}\right)^{q}\left(\left(t^{-}\right)^{2-q}\left\Vert u\right\Vert^{2}-\left(t^{-}\right)^{N-q}\int_M f\left\vert u\right\vert^{N}dv_g-\lambda\left\Vert u\right\Vert_{q}^{q}\right)\end{aligned}$$

and by (1.16) we deduce that

$$\Phi_\lambda(t^{-}u)=0$$

and also we get

$$\Phi_\lambda(t^{+}u)=0.$$

Moreover, we have

$$\left\langle\nabla\Phi_\lambda(t^{-}u),t^{-}u\right\rangle=2\left(t^{-}\right)^{2}\left\Vert u\right\Vert^{2}-N\left(t^{-}\right)^{N}\int_M f\left\vert u\right\vert^{N}dv_g-q\left(t^{-}\right)^{q}\lambda\left\Vert u\right\Vert_{q}^{q}$$

and taking account of (1.16) we infer that

$$\left\langle\nabla\Phi_\lambda(t^{-}u),t^{-}u\right\rangle=(2-q)\left(t^{-}\right)^{2}\left\Vert u\right\Vert^{2}-(N-q)\left(t^{-}\right)^{N}\int_M f\left\vert u\right\vert^{N}dv_g$$

and again by (1.16) we obtain

$$\begin{aligned}\left\langle\nabla\Phi_\lambda(t^{-}u),t^{-}u\right\rangle&=\left(t^{-}\right)^{1+q}\left((2-q)\left(t^{-}\right)^{1-q}\left\Vert u\right\Vert^{2}-(N-q)\left(t^{-}\right)^{N-q-1}\int_M f\left\vert u\right\vert^{N}dv_g\right)\\&=\left(t^{-}\right)^{1+q}E'\left(t^{-}\right)<0\end{aligned}$$

that means that $t^{-}u\in N_\lambda^{-}$. By similar procedure we get also $t^{+}u\in N_\lambda^{+}$.

2. Existence of a local minimizer for J_λ on N_λ^+ and N_λ^-

In this section we focus on the existence of a local minimum of J_λ on N_λ^+ and N_λ^- to do so we will be in need of the following Hardy-Sobolev inequality and Releich-Kondrakov embedding respectively whose proofs are given in ([6]). The weighted $L^p(M, \rho^\gamma)$ space will be the set of measurable functions u on M such that $\rho^\gamma |u|^p$ are integrable where $p \geq 1$ and γ are real numbers. We endow $L^p(M, \rho^\gamma)$ with the norm

$$\|u\|_{p,\rho} = \left(\int_M \rho^\gamma |u|^p \, dv_g\right)^{\frac{1}{p}}.$$

Theorem 4. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$ and p, q , γ are real numbers such that $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$. For any $\epsilon > 0$, there is $A(\epsilon, q, \gamma)$ such that for any $u \in H_2^2(M)$*

$$\|u\|_{p,\rho^\gamma}^2 \leq (1+\epsilon) K(n,2,\gamma)^2 \|\Delta_g u\|_2^2 + A(\epsilon, q, \gamma) \|u\|_2^2$$

where $K(n, 2, \gamma)$ is the optimal constant.

In case $\gamma = 0$, $K(n,2,0) = K(n,2) = K_o^{\frac{1}{2}}$ is the best constant in the Sobolev's embedding of $H_2^2(M)$ in $L^N(M)$ where $N = \frac{2n}{n-4}$.

Theorem 5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ and p, q, γ are real numbers satisfying $1 \leq q \leq p \leq \frac{nq}{n-2q}$, $\gamma < 0$ and $l = 1{,}2$.*

If $\frac{\gamma}{p} = n\ (\frac{1}{q} - \frac{1}{p}) - l$ then the inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is continuous. If $\frac{\gamma}{p} > n\ (\frac{1}{q} - \frac{1}{p}) - l$ then inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is compact.

The following variant of the Ekeland's variational principle will be also useful

Lemma 7. *If V is a Banach space and $J \in C^1(V, R)$ is bounded from below, then there exists a minimizing sequence (u_n) for J in V such that $J(u_n) \to \inf_V J$ and $E'(u_n) \to 0$ as $n \to \infty$.*

Lemma 8. *For any $\lambda \in (0,\ \lambda_\circ)$*

(i) There exists a minimizing sequence $(u_m)_m \subset N_\lambda$ such that $J_\lambda(u_m) = \alpha_\lambda + o(1)$ and $\nabla J_\lambda(u_m) = o(1)$

(ii) There exists a minimizing sequence $(u_m)_m \subset N_\lambda^-$ such that $J_\lambda(u_m) = \alpha_\lambda^- + o(1)$ and $\nabla J_\lambda(u_m) = o(1)$.

Proof. By Lemma 4 and the Enkland's variational principle (see 7) J_λ admits a Palais-Smale sequence at level α_λ in N_λ(the same is also true for (ii)). □

Now, we establish the existence of a local minimum for J_λ on N_λ^+

Theorem 6. *Let $\lambda \in (0,\ \lambda_\circ)$, and suppose that a sequence $(u_m)_m \subset N_\lambda^+$ fulfils*

$$\left\{ \begin{array}{c} J_\lambda(u_m) \leq c \\ \nabla J_\lambda(u_m) - \mu_m \nabla\Phi_\lambda(u_m) \to 0 \end{array} \right.$$

.

with

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (Max_{\ x\in M} f(x))^{\frac{n-4}{4}}}. \tag{C1}$$

Then the functional J_λ has a minimizer u^+ in N_λ^+ which satisfies

(i) $J_\lambda(u^+) = \alpha_\lambda^+ < 0$,

(ii) u^+ is a nontrivial solution of equation (1.3).

Proof. Let $(u_m)_m \subset N_\lambda^+$ be a Palais-Smale sequence for J_λ on N_λ i.e.

$$J_\lambda(u_m) = \alpha_\lambda + o(1) \text{ and } \nabla J_\lambda(u_m) = o(1) \text{ in } H_2^2(M)'.$$

Obviously

$$-\alpha_\lambda + o(1) \leq J_\lambda(u_m) - \frac{1}{q}\left\langle \nabla J_\lambda(u_m), u_m \right\rangle \leq \alpha_\lambda + o(1)$$

or

$$-\alpha_\lambda + o(1) \leq \left(\frac{N-2}{2N} - \frac{N-2}{Nq}\right) \|u_m\|^2 \leq \alpha_\lambda + o(1).$$

Hence

$$\alpha_\lambda \left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} + o(1) \leq \|u_m\|^2 \leq -\alpha_\lambda \left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} + o(1)$$

so the sequence $(u_m)_m$ is bounded in $H_2^2(M)$ and by the well known Sobolev's embedding, we get up to a subsequence that

$u_m \to u^+$ weakly in $H_2^2(M)$.

$u_m \to u^+$ strongly in $L^p(M)$ for $1 < p < N = \frac{2n}{n-4}$.

$\nabla u_m \to \nabla u^+$ strongly in $L^q(M)$ for $1 < q < 2^* = \frac{2n}{n-2}$.

$u_m \to u^+$ a.e in M.

Put

$$w_m := u_m - u^+$$

by Brézis-Lieb Lemma (see [9]), we obtain

$$\|\Delta_g u_m\|_2^2 - \left\|\Delta_g u^+\right\|_2^2 = \|\Delta_g w_m\|_2^2 + o(1)$$

and

$$\int_M f(x) \left(|u_m|^N - \left|u^+\right|^N\right) dv(g) = \int_M f(x) \left|w_m\right|^N dv(g) + o(1)$$

Now since $\sigma \in (0,2)$ and $\mu \in (0,4)$, by Theorem 5 we infer that $\nabla u_m \to \nabla u^+$ strongly in $L^2(M, \rho^{-\sigma})$ and $u_m \to u^+$ strongly in $L^2(M, \rho^{-\mu})$.First, we prove that $u^+ \in N_\lambda$.

Taking into account of the strong convergences of $\nabla u_m \to \nabla u^+$ in $L^2\left(M, \rho^{-\sigma}\right)$ and $u_m \to u^+$ in $L^2\left(M, \rho^{-\mu}\right)$, we obtain

$$J_\lambda(u_m) - J_\lambda(u^+)$$

$$= \frac{1}{2}\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 - \frac{1}{N}\int_M f(x)\left|u_m - u^+\right|^N dv(g) + o(1). \tag{2.1}$$

Since $u_m - u^+ \to 0$ weakly in $H_2^2(M)$, we test by $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$ and get

$$\left\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u^+), u_m - u^+ \right\rangle =$$

$$\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 - \int_M f(x)\left|u_m - u^+\right|^N dv(g) = o(1). \tag{2.2}$$

So by (2.2), we obtain

$$J_\lambda(u_m) - J_\lambda(u^+) = \frac{1}{2}\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 - \frac{1}{N}\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u^+) = \frac{2}{n}\left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 + o(1).$$

By Sobolev's inequality, we have for all $u \in H_2^2(M)$

$$\|u\|_N^2 \le (1+\varepsilon)K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2 dv(g) + A_\varepsilon \int_M u^2 dv(g)$$

We test the Sobolev's inequality by $u_m - u$, to get

$$\left\|u_m - u^+\right\|_N^2 \le (1+\varepsilon)K_\circ \int_M \left(\Delta_g(u_m - u^+)\right)^2 dv(g) + o(1). \tag{2.3}$$

Then (2.3) implies that

$$\int_M f(x)\left|u_m - u^+\right|^N dv(g) \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m - u^+)\right\|_2^N + o(1)$$

and by (2.2) one writes

$$o(1) \ge \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m - u^+)\right\|_2^N + o(1).$$

or in another words

$$o(1) \ge \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^{N-2}) + o(1).$$

Hence if

$$\limsup_{m\to+\infty} \left\|\Delta_g\left(u_m - u^+\right)\right\|_2^{N-2} < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x\in M} f(x)}$$

we get

$$\frac{2}{n}\int_M \left(\Delta_g(u_m - u^+)\right)^2 dv(g) < c.$$

and since by assumption

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (Max_{\ x\in M} f(x))^{\frac{n-4}{4}}}$$

we deduce that

$$\int_M \left(\Delta_g(u_m - u^+)\right)^2 dv(g) < \frac{1}{K_\circ^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n-4}{4}}}.$$

Hence

$$o(1) \geq \left\|\Delta_g(u_m - u^+)\right\|_2^2 \underbrace{(1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m - u^+)\right\|_2^{N-2})}_{>0} + o(1)$$

or

$$\left\|\Delta_g(u_m - u^+)\right\|_2^2 = o(1)$$

i.e. $u_m \to u^+$ converges strongly in $H_2^2(M)$.
Obviously $u^+ \in N_\lambda$. We claim that $u^+ \in N_\lambda^+$ since it is not the case $u^+ \in N_\lambda^-$, thus $\langle \nabla J_\lambda(u^+), u^+\rangle = 0$ and $\langle \nabla \Phi_\lambda(u^+), u^+\rangle < 0$, which implies that $J_\lambda(u^+) > 0$, contradiction.
Then,

$$J_\lambda(u^+) = \alpha_\lambda^+ = \alpha_\lambda < 0.$$

Now, we want to prove that u^+ is a trivial solution to equation (1.3) but this follows from Lemma 2 since in that case u^+ is a global minimizer of J_λ in $H_2^2(M)$. $\square$

Theorem 7. *Let $\lambda \in (0, \lambda_\circ)$ and suppose that a sequence $(u_m)_m \subset N_\lambda^-$ fulfils*

$$\begin{cases} J_\lambda(u_m) \leq c \\ \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0 \end{cases}$$

.

with

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} (Max\ _{x\in M} f(x))^{\frac{n-4}{4}}}. \tag{2.4}$$

Then the functional J_λ has a minimizer u^- in N_λ^- and it satisfies
(i) $J_\lambda(u^-) = \alpha_\lambda^- > 0$,
(ii) u^- is a nontrivial solution of equation (1.1).

Proof. The proof is similar to that of Theorem 6, so we omit it. $\square$

Remark 1. *The nontrivial solutions u^+ and u^- of equation (1.1) given by Theorem 6 and Theorem7 are distinct since $u^+ \in N_\lambda^+$, $u^- \in N_\lambda^-$ and $N_\lambda^+ \cap N_\lambda^- = \emptyset$.*

3. The sharp case $\sigma = 2$ and $\mu = 4$

By section four, for any $\sigma \in (0,2)$ and $\mu \in (0,4)$, there is a weak solution $u_{\sigma,\mu}^+ \in N_\lambda^+$ (resp. $u_{\sigma,\mu}^- \in N_\lambda^-$) of equation (1.3). Now we are going to show

that the sequence $\left(u^+_{\sigma,\mu}\right)_{\sigma,\mu}$ and $\left(u^-_{\sigma,\mu}\right)_{\sigma,\mu}$are bounded in $H^2_2(M)$. First of all we have

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{1}{2}\left\|u_{\sigma,\mu}\right\|^2 - \frac{1}{N}\int_M f(x)\left|u_{\sigma,\mu}\right|^N dv_g - \frac{1}{q}\lambda\int_M \left|u_{\sigma,\mu}\right|^q dv_g$$

and since $u_{\sigma,\mu} \in N_\lambda$, we infer that

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{N-2}{2N}\left\|u_{\sigma,\mu}\right\|^2 - \lambda\frac{N-q}{Nq}\int_M \left|u_{\sigma,\mu}\right|^q dv_g.$$

For a smooth function a on M, denotes by $a^- = \min\left(0, \min_{x\in M}(a(x)\right)$. Let $K(n,2,\sigma)$ the best constant and $A(\varepsilon,\sigma)$ the constants in the Hardy-Sobolev inequality.

Denote by $(u^+_{\sigma_m,\mu_m})_m$ a countable subsequence of the sequence $(u^+_{\sigma,\mu})_{\sigma,\mu}$ given above.

Theorem 8. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $(u^+_m)_m = (u^+_{\sigma_m,\mu_m})_m$ be a sequence in N^+_λ such that*

$$\left\{\begin{array}{c} J_{\lambda,\sigma,\mu}(u^+_m) \leq c_{\sigma,\mu} \\ \nabla J_\lambda(u^+_m) - \mu_{\sigma_m,\mu_m}\nabla\Phi_\lambda(u^+_m) \to 0 \end{array}\right. .$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n\ K(n,2)^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max\left(K(n,2,\sigma), A(\varepsilon,\sigma)\right) + b^- \max\left(K(n,2,\mu), A(\varepsilon,\mu)\right) > 0$$

then the equation

$$\Delta^2 u - \nabla^\mu\left(\frac{a}{\rho^2}\nabla_\mu u\right) + \frac{bu}{\rho^4} = f\left|u\right|^{N-2}u + \lambda\left|u\right|^{q-2}u \tag{3.1}$$

has a non trivial solution $u^+ \in N^+_\lambda$ in the distribution .

Proof. Let $(u^+_m)_m = \left(u^+_{\sigma_m;\mu_m}\right)_m \subset N^+_{\lambda,\sigma,\mu}$,

$$J_{\lambda,\sigma,\mu}(u^+_m) = \frac{N-2}{2N}\left\|u^+_m\right\|^2 - \lambda\frac{N-q}{Nq}\int_M \left|u^+_m\right|^q dv_g$$

As in proof of Theorem 6, we get

$$J_{\lambda,\sigma,\mu}(u^+_m) \geq$$

$$\left\|u^+_m\right\|^2\left(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}_{\sigma,\mu}V(M)^{1-\frac{q}{N}}\left(\max((1+\varepsilon)K(n,2), A_\varepsilon)\right)^{\frac{q}{2}}\tau^{q-2}\right) > 0$$

where $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{\frac{q}{2}}_{\sigma,\mu}}{V(M)^{1-\frac{q}{N}}\left(\max((1+\varepsilon)K(n,2),A_\varepsilon)\right)^{\frac{q}{2}}\tau^{q-2}} = \lambda^+_{o,\sigma,\mu}$ and $\Lambda_{\sigma,\mu}$ is the coercivity's constant (which depends on σ and μ)

First we claim that

$$\lim_{(\sigma,\mu)\to(2^-,4^-)}\inf\Lambda_{\sigma,\mu} > 0.$$

Indeed, if $\nu_{1,\sigma,\mu}$ denotes the first nonzero eigenvalue of the operator $u \to P_g(u) = \Delta_g^2 u - div\left(\frac{a}{\rho^\sigma}\nabla_g u\right) + \frac{bu}{\rho^\mu}$, then clearly $\Lambda_{\sigma,\mu} \geq \nu_{1,\sigma,\mu}$. Suppose by absurd that $\lim_{(\sigma,\mu)\to(2^-,4^-)}\inf \Lambda_{\sigma,\mu} = 0$, then $\liminf_{(\sigma,\mu)\to(2^-,4^-)} \nu_{1,\sigma,\mu} = 0$. Independently, if $u_{\sigma,\mu}$ is the corresponding eigenfunction to $\nu_{1,\sigma,\mu}$ we have

$$\nu_{1,\sigma,\mu} = \|\Delta u\|_2^2 + \int_M \frac{a\,|\nabla u|^2}{\rho^\sigma} dv_g + \int_M \frac{bu^2}{\rho^\mu} dv_g$$

$$\geq \|\Delta u\|_2^2 + a^- \int \frac{|\nabla u|^2}{\rho^\sigma} dv_g + b^- \int_M \frac{u^2}{\rho^\mu} dv_g \tag{3.2}$$

where $a^- = \min\left(0, \min_{x\in M} a(x)\right)$ and $b^- = \min\left(0, \min_{x\in M} b(x)\right)$. The Hardy- Sobolev's inequality leads to

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C(\|\nabla\,|\nabla u|\|^2 + \|\nabla u\|^2)$$

and since

$$\|\nabla\,|\nabla u|\|^2 \leq \left\|\nabla^2 u\right\|^2 \leq \|\Delta u\|^2 + \beta\,\|\nabla u\|^2$$

where $\beta > 0$ is a constant and it is well known that for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u\|^2 \leq \varepsilon\,\|\Delta u\|^2 + c\,\|u\|^2 .$$

Hence

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq C\,(1+\varepsilon)\,\|\Delta u\|^2 + A\,(\varepsilon)\,\|u\|^2 \tag{3.3}$$

Now if $K(n, 2, \sigma)$ denotes the best constant in inequality (3.3) we get for any $\varepsilon > 0$

$$\int_M \frac{|\nabla u|^2}{\rho^\sigma} dv_g \leq \left(K(n,2,\sigma)^2 + \varepsilon\right)\|\Delta u\|^2 + A\,(\varepsilon,\sigma)\,\|u\|^2 . \tag{3.4}$$

By the inequalities (2.3), (3.2) and (3.4), we have

$$\nu_{1,\sigma,\mu} \geq \left(1 + a^- \max\left(K(n,2,\sigma), A\,(\varepsilon,\sigma)\right) + b^- \max\left(K(n,2,\mu), A\,(\varepsilon,\mu)\right)\right)$$
$$\times\left(\|\Delta u_{\sigma,\mu}\|^2 + \|u_{\sigma,\mu}\|^2\right).$$

So if

$$1 + a^- \max\left(K(n,2,\sigma), A\,(\varepsilon,\sigma)\right) + b^- \max\left(K(n,2,\mu), A\,(\varepsilon,\mu)\right) > 0$$

then we get $\lim_{\sigma,\mu}\left(u_{\sigma,\mu}\right) = 0$ and $\|u_{\sigma,\mu}\| = 1$ a contradiction. Denote by

$$\Lambda = \liminf_{\sigma,\mu} \Lambda_{\sigma,\mu}.$$

The same arguments as in the proof of Theorem6 we obtain that

$$a_\lambda^+\left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} \leq \left\|u_m^+\right\|_{\sigma,\mu}^2 \leq -a_\lambda^+\left(\frac{N-2}{Nq} - \frac{N-2}{2N}\right)^{-1} + o(1)$$

where

$$\left\|u^{+}\right\|_{\sigma,\mu}^{2}=\left\|\Delta u^{+}\right\|_{2}^{2}-\int_{M}\left(a(x)\frac{\left|\nabla_{g}u^{+}\right|^{2}}{\rho^{\sigma}}+\frac{b(x)}{\rho^{\mu}}\left(u^{+}\right)^{2}\right)dv(g).$$

It is easily seen by the Lebesque'dominated convergence theorem that $\left\|u^{+}\right\|_{\sigma,\mu}$ goes to $\left\|u^{+}\right\|_{2,4}$ as $(\sigma,\mu)\to(2,4)$.

Now by reflexivity of $H_2^2(M)$ and the compactness of the embedding $H_2^2(M)\subset H_p^k(M)$ ($k=0,1$; $p<N$), we obtain up to a subsequence we have:

$u^{+}_{\sigma_m,\mu_m}\to u^{+}$ weakly in $H_2^2(M)$
$u^{+}_{\sigma_m,\mu_m}\to u^{+}$ strongly in $L^p(M)$, $p<N$
$\nabla u^{+}_{\sigma_m,\mu_m}\to \nabla u^{+}$ strongly in $L^p(M)$, $p<2^{*}=\frac{2n}{n-2}$
$u^{+}_{\sigma_m,\mu_m}\to u^{+}$ a.e. in M.

For brevity we let $u_m^{+}=u^{+}_{\sigma_m,\mu_m}$

The Brézis-Lieb lemma allows us to write

$$\int_M\left(\Delta_g u_m^{+}\right)^2dv_g=\int_M\left(\Delta_g u^{+}\right)^2dv_g+\int_M\left(\Delta_g(u_m^{+}-u^{+})\right)^2dv_g+o(1)$$

and also

$$\int_M f(x)\left|u_m^{+}\right|^N dv_g=\int_M f(x)\left|u^{+}\right|^N dv_g+\int_M f(x)\left|u_m^{+}-u^{+}\right|^N dv_g+o(1).$$

Now by the boundedness of the sequence $(u_m^{+})_m$, we have that $u_m^{+}\to u^{+}$ weakly in $H_2^2(M)$, $\nabla u_m^{+}\to\nabla u^{+}$ weakly in $L^2(M,\rho^{-2})$ and $u_m^{+}\to u^{+}$ weakly in $L^2(M,\rho^{-4})$ i.e. for any $\varphi\in L^2(M)$If $\delta\in(0,\delta(M))$ then we obtain for every $\varphi\in H_2^2(M)$

(3.5)

$$\int_M\frac{b(x)}{\rho^{\mu_m}}\left(u_m^{+}\right)^2\varphi dv(g)=\int_{B_P(\delta)}\frac{b(x)}{\rho^{\mu_m}}\left(u_m^{+}\right)^2dv(g)+\int_{M-B_P(\delta)}\frac{b(x)}{\rho^{\mu_m}}\left(u_m^{+}\right)^2dv(g)$$

and

$$\int_M\frac{b(x)}{\rho^{\delta_m}}\left(u^{+}\right)^2dv(g)=\int_M\frac{b(x)}{\rho^{4}}\left(u^{+}\right)^2dv(g)+o(1)\quad\text{when}\quad\delta_m\to4^{-}.$$

Now the fact $u_m^{+}\to u^{+}$ weakly in $H_2^2(M)$, $\nabla u_m^{+}\to\nabla u^{+}$ weakly in $L^2(M,\rho^{-2})$ and $u_m^{+}\to u^{+}$ weakly in $L^2(M,\rho^{-4})$ expresses as: for all $\varphi\in L^2(M)$:

$$\int_M\frac{a(x)}{\rho^2}\nabla u_m^{+}\nabla\varphi dv(g)=\int_M\frac{a(x)}{\rho^2}\nabla u^{+}\nabla\varphi dv(g)+o(1)$$

and

$$\int_M\frac{b(x)}{\rho^4}u_m^{+}\varphi dv(g)=\int_M\frac{b(x)}{\rho^4}u^{+}\varphi dv(g)+o(1).$$

Consequently u^{+} is a weak solution to equation (3.1).

Since $u_m^{+}\to u^{+}$weakly in $H_2^2(M)$, we have for all $\phi\in L^2(M)$

$$\int_M\left(u_m^{+}-u^{+}\right)\Delta_g^2\phi dv(g)=o(1)$$

then,

$$\int_M u_m^+ \Delta_g^2 \phi dv(g) = \int_M \Delta_g \phi \Delta_g u^+ dv(g) + o(1).$$

For the second integral, we obtain

$$\int_M \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m^+ - \frac{a(x)}{\rho^2} \nabla_g u^+ \right) \nabla \phi dv(g) =$$

$$\int_M \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m^+ + \frac{a(x)}{\rho^2} \left(\nabla_g u_m^+ - \nabla_g u_m^+ \right) - \frac{a(x)}{\rho^2} \nabla_g u^+ \right) \nabla \phi dv(g).$$

Consequently

$$\left| \int_M \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m^+ - \frac{a(x)}{\rho^2} \nabla_g u^+ \right) \nabla \phi dv(g) \right| \leq$$

$$\left| \int_M \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m^+ - \frac{a(x)}{\rho^2} \nabla_g u_m^+ \right) \nabla \phi dv(g) \right| + \left| \int_M \left(\frac{a(x)}{\rho^2} \nabla_g u_m^+ - \frac{a(x)}{\rho^2} \nabla_g u^+ \right) \nabla \phi dv(g) \right|$$

$$\leq \left| \int_M \frac{a(x)}{\rho^2} \nabla_g \left(u_m^+ - u^+ \right) \nabla \phi dv(g) \right| + \int_M \left| a(x) \nabla \phi \nabla_g u_m^+ \right| \left| \frac{1}{\rho^{\sigma_m}} - \frac{1}{\rho^2} \right| dv(g).$$

By the weak convergence in $L^2(M, \rho^{-2})$ and the dominated Lebesgue's convergence theorem, we obtain that

$$\int_M \left(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m^+ - \frac{a(x)}{\rho^2} \nabla_g u^+ \right) \nabla \phi dv(g) = o(1).$$

The third integral splits as

$$\int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m^+ - \frac{b(x)}{\rho^4} u^+ \right) \phi dv(g) =$$

$$\int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m^+ - \frac{b(x)}{\rho^4} u_m^+ + \frac{b(x)}{\rho^4} u_m^+ - \frac{b(x)}{\rho^4} u^+ \right) \phi dv(g)$$

so

$$\left| \int_M \left(\frac{b(x)}{\rho^{\mu_m}} u_m^+ - \frac{b(x)}{\rho^4} u^+ \right) \phi dv(g) \right|$$

$$\leq \int_M \left| b(x) \phi u_m \right| \left| \frac{1}{\rho^{\mu_m}} - \frac{1}{\rho^4} \right| dv(g) + \left| \int_M \frac{b(x)}{\rho^4} \left(u_m^+ - u^+ \right) \phi dv(g) \right|$$

and by the same arguments, we obtain that

$$\int_M \left(\frac{b(x)}{\rho^{\delta_m}} u_m^+ - \frac{b(x)}{\rho^4} u^+ \right) \phi dv(g) = o(1) \quad .$$

It remains to show that $\mu_m \to 0$ as $m \to +\infty$ and $u_m^+ \to u^+$ strongly in $H_2^2(M)$ but this is the same as in the proof of Theorem 7.

Consequently u^+ is a nontrivial solution in N_λ^+ of equation .

$\square$

4. Test Functions

To give the proof of the main result, we consider a normal geodesic coordinate system centred at x_o. Let $S_{x_o}(\rho)$ the geodesic sphere centred at x_o and of radius ρ strictly less that the injectivity radius d. Let dv_h be the volume element of the $n-1$-dimensional Euclidean unit sphere S^{n-1} endowed with its canonical metric and put

$$G(\rho) = \frac{1}{\omega_{n-1}} \int\limits_{S(\rho)} \sqrt{|g(x)|}\, dv_h$$

where ω_{n-1} is the volume of S^{n-1} and $|g(x)|$ the determinant of the Riemannian metric g. The Taylor's expansion of $G(\rho)$ in a neighborhood of x_o expresses as

$$G(\rho) = 1 - \frac{S_g(x_\circ)}{6n}\rho^2 + o(\rho^2)$$

where $S_g(x_\circ)$ is the scalar curvature of M at $x_\circ$.
If $B(x_\circ, \delta)$ is the geodesic ball centred at $x_\circ$ and of radius δ such that $0 < 2\delta < d$, we consider the following cutoff smooth function η on M

$$\eta(x) = \begin{cases} 1 & \text{on } B(x_o, \delta) \\ 0 & \text{on } M - B(x_o, 2\delta) \end{cases} .$$

Define the following radial function

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{8}} \frac{\eta(\rho)}{\left((\rho\theta)^2 + \epsilon^2\right)^{\frac{n-4}{2}}}$$

with

$$\theta = \left(1 + \left\|\frac{a}{\rho^\sigma}\right\|_r + \left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{1}{n}}$$

where $\rho = d(x_o, x)$ is the distance from x_o to x and $f(x_\circ) = \max_{x\in M} f(x)$.
We need also the following integrals: for any real positive numbers p, q such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt$$

which fulfill the following relations

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q .$$

5. Application to compact Riemannian manifolds of dimension $n > 6$

Theorem 9. *Let (M, g) be a compact Riemannian manifold of dimension $n > 6$. Suppose that at a point x_o where f attains its maximum the following*

condition

$$\frac{\Delta f(x_o)}{f\left(x_o\right)}<\frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{\left(n^2-4\right)\left(n-4\right)\left(n-6\right)}\frac{1}{\left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{4}{n}}}-1\right)S_g\left(x_o\right)$$

holds . Then the equation (1.1) has at least two non trivial solutions.

Proof. The proof of Theorem 2 reduces to show that the condition (C1) of Theorem 6 which is the same condition (2.4) of Theorem **??** is satisfied and since at the end of section 1, we have shown that for a given $u\in H_2^2\left(M\right)$ there exist two real numbers $t^->0$ and $t^+>0$ such that $t^-u\in N_\lambda^-$ and $t^+u\in N_\lambda^+$ for sufficiently small λ, so it suffices to show that

$$\sup_{t>0}J_\lambda\left(tu_\epsilon\right)<\frac{1}{K_\circ^{\frac{n}{4}}\left(\max_{x\in M}f(x)\right)^{\frac{n}{4}-1}}.$$

The expression of $\int_M f(x)\left|u_\epsilon(x)\right|^N dv_g$ is well known (see for example [10]) and is given in case $n>6$ by

$$\int_M f(x)\left|u_\epsilon(x)\right|^N dv_g=\frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1-\left(\frac{\Delta f(x_\circ)}{2(n-2)f(x_\circ)}+\frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2+o(\epsilon^2)\right).$$

The following estimation is computed in [7] and is given by

$$\int_M\frac{a(x)}{\rho^\sigma}\left|\nabla u_\epsilon\right|^2dv_g\le$$

$$2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left\|\frac{a}{\rho^\sigma}\right\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}$$
$$\times\left(I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}.\frac{r-1}{r}}+o(\epsilon^2)\right).$$

Letting

$$A=K_\circ^{\frac{n}{4}}\frac{(n-4)^{\frac{n}{4}+1}\times\left(\omega_{n-1}\right)^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(n(n^2-4))^{\frac{n-4}{4}}\left(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \tag{5.1}$$

we obtain

$$\int_M a(x)\left|\nabla u_\epsilon\right|^2dv_g\le\epsilon^{2-\frac{n}{r}}\theta^{-n\frac{r}{r-1}}\frac{A}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left\|\frac{a}{\rho^\sigma}\right\|_r\left(1+o(\epsilon^2)\right).$$

Also the estimation of the third term of J_λ is computed in [7] as

$$\int_M\frac{b(x)}{\rho^\mu}u_\epsilon^2dv_g\le\left\|b\right\|_s\left(\frac{(n-4)n(n^2-4)}{f(x_\circ)}\right)^{\frac{n-4}{4}}\left(\frac{\omega_{n-1}}{2}\right)^{\frac{s-1}{s}}\epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}$$
$$\times\left(\left(I_{\frac{(n-4)s}{(s-1)}}^{\frac{n}{2}}\right)^{\frac{s-1}{s}}+o(\epsilon^2)\right)$$

Putting

$$B = K_\circ^{\frac{n}{4}}((n-4)n(n^2-4))^{\frac{n-4}{4}}(\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}}\left(I_{\frac{(n-4)s}{(s-1)}}^{\frac{n}{2}}\right)^{\frac{s-1}{s}} \tag{5.2}$$

we get

$$\int_M b(x)u_\epsilon^2 dv_g \le \epsilon^{4-\frac{n}{s}}\theta^{-n\frac{s}{s-1}}\frac{\left\|\frac{b}{\rho^\mu}\right\|_s B}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1+o(\epsilon^2)\right).$$

The computation of $\int_M (\Delta u_\epsilon)^2 dv_g$ is well known see for example ([10]) and is given by

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}\left(1-\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right).$$

Resuming we get

$$\int_M (\Delta u_\epsilon)^2 - a(x)\left|\nabla u_\epsilon\right|^2 + b(x)u_\epsilon^2 dv_g \le \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}f(x_\circ)^{\frac{n-4}{4}}}\times$$

$$\left(1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\left\|\frac{a}{\rho^\sigma}\right\|_r+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\left\|\frac{b}{\rho^\mu}\right\|_s-\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_\circ)\epsilon^2+o(\epsilon^2)\right).$$

Now, we have

$$J_\lambda(tu_\epsilon) \le J_o(tu_\epsilon) = \frac{t^2}{2}\left\|u_\epsilon\right\|^2 - \frac{t^N}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv_g$$

$$\le \frac{\theta^{-n}}{K_\circ^{\frac{n}{4}}f(x_\circ)^{\frac{n-4}{4}}}\left\{\frac{1}{2}t^2\left(1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\left\|\frac{a}{\rho^\sigma}\right\|_r+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\left\|\frac{b}{\rho^\mu}\right\|_s\right)-\frac{t^N}{N}\right.$$

$$\left.+\left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)}+\frac{S_g(x_o)}{6(n-1)}\right)\frac{t^N}{N}-\frac{1}{2}t^2\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_o)\right]\epsilon^2\right\}$$

$$+o(\epsilon^2)$$

and letting ϵ small enough so that

$$1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\left\|\frac{a}{\rho^\sigma}\right\|_r+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\left\|\frac{b}{\rho^\mu}\right\|_s \le \left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{4}{n}}$$

and since the function $\varphi(t) = \alpha\frac{t^2}{2}-\frac{t^N}{N}$, with $\alpha > 0$ and $t > 0$, attains its maximum at $t_o = \alpha^{\frac{1}{N-2}}$ and

$$\varphi(t_o) = \frac{2}{n}\alpha^{\frac{n}{4}}.$$

Consequently, we get

$$J_\lambda(tu_\epsilon) \le \frac{2\theta^{-n}}{nK_\circ^{\frac{n}{4}}f(x_\circ)^{\frac{n-4}{4}}}\left\{1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right.$$

$$\left.+\left[\left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)}+\frac{S_g(x_o)}{6(n-1)}\right)\frac{t_o^N}{N}-\frac{1}{2}t_o^2\frac{n^2+4n-20}{6(n^2-4)(n-6)}S_g(x_o)\right]\epsilon^2\right\}$$

$$+o\left(\epsilon^{2}\right).$$

Taking account of the value of θ and putting

$$R(t)=\left(\frac{\Delta f(x_o)}{2\left(n-2\right)f(x_o)}+\frac{S_g\left(x_o\right)}{6\left(n-1\right)}\right)\frac{t^N}{N}-\frac{1}{2}\frac{n^2+4n-20}{6\left(n^2-4\right)\left(n-6\right)}S_g\left(x_o\right)t^2$$

we obtain

$$\sup_{t\geq 0} J_\lambda\left(tu_\epsilon\right)<\frac{2}{nK_\circ^{\frac{n}{4}}\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1}}$$

provided that $R(t_o)<0$ i.e.

$$\frac{\Delta f(x_o)}{f\left(x_o\right)}<\left(\frac{n\left(n^2+4n-20\right)}{3\left(n+2\right)\left(n-4\right)\left(n-6\right)}\frac{1}{\left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{4}{n}}}-\frac{n-2}{3\left(n-1\right)}\right)S_g\left(x_o\right).$$

Which completes the proof. $\square$

6. Application to compact Riemannian manifolds of dimension $n=6$

Theorem 10. *In case $n=6$, we suppose that at a point x_o where f attains its maximum $S_g\left(x_o\right)>0$. Then the equation (1.1) has at least two distinct non trivial solutions in the distribution sense..*

Proof. In case $n=6$ the only term whose expression differs from the case $n>6$ is the first term of J_λ and is given (see for example [10]) by

$$\int_M\left(\Delta u_\epsilon\right)^2 dv_g=\frac{\theta^n}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1-\frac{2\left(n-4\right)}{n^2(n^2-4)I_n^{\frac{n}{2}-1}}S_g(x_\circ)\epsilon^2\log\left(\frac{1}{\epsilon^2}\right)+O(\epsilon^2)\right).$$

Letting ϵ so that

$$1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\left\|\frac{a}{\rho^\sigma}\right\|_r+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\left\|\frac{b}{\rho^\mu}\right\|_s\leq\left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right)^{\frac{4}{n}}$$

where A and B are given by (5.1) and (5.2), we get

$$J_\lambda\left(u_\epsilon\right)\leq\frac{1}{2}\left\|u_\epsilon\right\|^2-\frac{1}{N}\int_M f(x)\left|u_\epsilon(x)\right|^N dv_g$$

$$\leq\frac{\theta^n}{K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left[\frac{t^2}{2}\left(1+\left\|\frac{a}{\rho^\sigma}\right\|_r+\left\|\frac{b}{\rho^\mu}\right\|_s\right)^{1-\frac{4}{n}}-\frac{t^N}{N}\right.$$
$$\left.-\frac{n-4}{n^2\left(n^2-4\right)I_n^{\frac{n}{2}-1}}\theta^{-2}S_g(x_\circ)t^2\epsilon^2\log\left(\frac{1}{\epsilon^2}\right)\right]+O(\epsilon^2).$$

As in the case $n>6$ we infer that

$$\max_{t\geq 0} J_\lambda\left(tu_\epsilon\right)<\frac{2}{n\ K_\circ^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

provided that

$$S_g(x_\circ) > 0.$$

Which achieves the proof. $\square$

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